RADCOM TECHNICAL FEATURE

Complex Numbers Explained

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I WROTE AN ARTICLE ENTITLED 'The "j" Operator and Impedance' in *RadCom*, June 1990, and this received a favourable response. I hope it showed readers the usefulness of this operator. The following article develops the use of the j operator, and enables users to manipulate circuit equations they may derive or come across. Various problems are presented and small computer programs provided to perform the analysis. Some calculators can cope with complex numbers and switch between polar and cartesian coordinates.

The computer programs in the listings have been tested on an IBM-PC (GWBASIC) and a BBC computer. It should be easy to convert them to other dialects of BASIC.

THINK OF A NUMBER

LET'S START WITH a little revision. In mathematics, numbers can be classified into three basic types – real numbers, imaginary numbers and complex numbers. Real numbers are numbers such as 3 (an integer), rational numbers (fractions) and irrational numbers such as π and $\sqrt{2}$. An imaginary number is a real number but prefixed by the operator j (eg j5). A complex number is a combination of these two, eg 3.5 + j7.6.

PROPERTIES OF THE J OPERATOR

IT WAS SHOWN IN the June 1990 article that j can be used for a term at 90° (or quadrature) to another term – eg in calculating impedance. Its use, however, is not restricted to electrical theory, though this is a major application.

If a quantity of value A is taken, when multiplied by j it will assume a value of jA and be in the direction shown in **Fig 1**, ie a rotation of 90° anti-clockwise. A further multiplication by j rotates jA by a further 90°, this is now in the opposite direction of A and hence equals –A. The implication of this is that $j \times j = -1$. Further multiplication by j provides –jA (Fig 1)

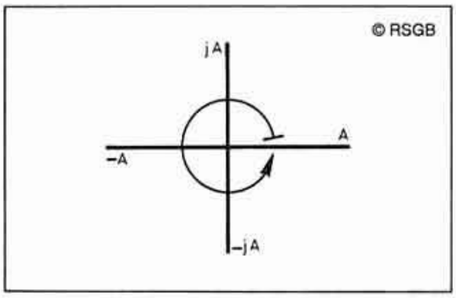

Fig 1: Graphical representation of impedance.

and a further rotation of j brings one back to A, i.e. $-j \times j = +1$.

The fact that $j \times j = -1$ gives the implication that $j = \sqrt{-1}$ – the concept of the square root of a negative number! Some readers may recall being told at school that you cannot take the square root of a negative number. However, this is a convenient mathematical concept for dealing with complex numbers.

Taking j as the square root of –1, then the following are true:

$$j = \sqrt{-1}$$
$$j \times j = -1$$
$$j \times j \times j = -1 \times j = -j$$
$$j \times j \times j \times j = -1 \times -1 = +1$$

Next, we'll consider the concept of vectors – a quantity that has both magnitude and direction – eg Force. In many engineering problems a vector is split into two components at right angles (90°) to each other – the x and y or real and imaginary components or whatever they are called. This often makes further analysis easier and it is also where the j operator can help. Remember that 90° phase shifts come into electrical theory when capacitive and inductive components are introduced into circuits.

SOME BASIC RULES

THERE ARE VARIOUS rules for using complex numbers, which are necessary for calculations. These are:

ADDITION

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$
$$\text{eg } (3 + j4) + (5 - j8) = 8 - j4$$

SUBTRACTION

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$
$$\text{eg } (5 + j6) - (3 + j2) = 2 + j4$$

COMPLEX CONJUGATE

The complex conjugate of a complex number is defined by:

Number	Complex Conjugate
a + jb	a – jb
a – jb	a + jb

ie the sign of the j term has been reversed.

If z is a complex number, the conjugate is normally written as z^*. One property of adding a complex number to its conjugate is that the result is a real number.

$$\text{eg } (3 + j6) + (3 - j6) = 6$$

(note that there is no j term)

MULTIPLICATION

$$(a + jb)(c + jd) = ac + jad + jbc + j^2bd$$

However, $j^2 = -1$, therefore this reduces to:

$$(ac - bd) + j(ad + bc)$$
$$\text{eg } (5 + j7)(2 + j3) = -11 + j29$$

An interesting result occurs when a complex number is multiplied by its complex conjugate:

$$(a + jb)(a - jb) = a^2 + jab - jab + b^2 = a^2 + b^2$$

Listing 1 will multiply two complex numbers together.

DIVISION

This is somewhat more arduous and requires the complex number to be only on the top line. The form of a division is:

$$\frac{(a + jb)}{(c + jd)}$$

To proceed, multiply the bottom line by its complex conjugate, to balance this the top line must also be multiplied by this number.

$$\frac{(a + jb)}{(c + jd)} \times \frac{(c - jd)}{(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{(c^2 + d^2)} = \frac{(ac + bd)}{c^2 + d^2} + j\frac{(bc - ad)}{c^2 + d^2}$$

eg
$$\frac{(2 + j3)}{(4 + j2)} = 0.7 + j0.4$$

Listing 2 will divide two complex numbers – try it!

CARTESIAN AND POLAR COORDINATES

THESE TWO FANCY SOUNDING names are the terms used for two different ways of defining a point. The cartesian coordinates system uses distances at right angles to specify a position – for instance, three steps right, four steps forward. The polar coordinate system uses a distance and an angle – for instance walk five steps at a bearing of 53°.

These two methods are equivalent and you would arrive at the same position. This situation is shown on **Fig 2**. By considering these two plots it can be seen that either set of coordinates is adequate and there must be a way of converting from one to the other.

Putting it all on a mathematical footing, the cartesian system is equivalent to the familiar x – y graphs, the other form is a magnitude and an angle – an example is the radiation pattern of an antenna. The general forms are depicted on **Fig 3**, the relationship between the two forms being given by:

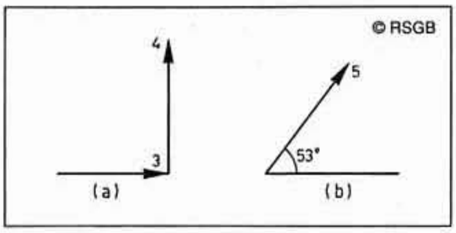

Fig 2: Cartesian and Polar coordinates.

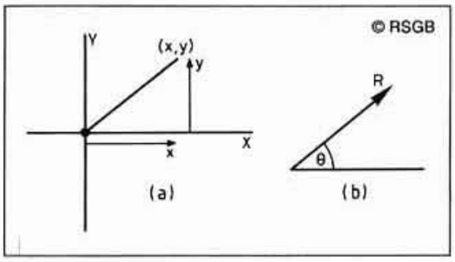

Fig 3: Resultant magnitude and angle.

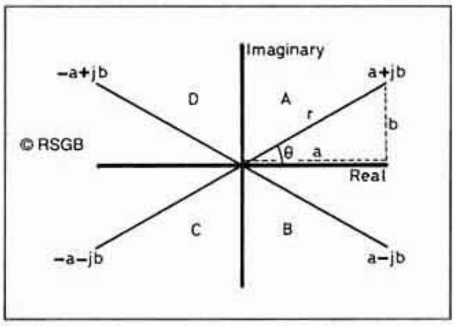

Fig 4: Complex number in graphical form.

$$x = R \cos\theta$$
$$y = R \sin\theta$$

Conversely:

$$\tan\theta = x/y \text{ or } \theta = \tan^{-1}(x/y)$$
$$R^2 = x^2 + y^2$$

Note that if the values of x and y are known, then calculation of the angle requires the use of the inverse tan function ($\tan^{-1}$). This is available on many calculators or in BASIC as the ATN function. **Listings 3 and 4** provide simple programs that will convert between the two coordinate systems. Having established these relationships, the next step is to relate it to complex numbers.

MODULUS AND ARGUMENT OF A COMPLEX NUMBER

THE MODULUS OF A complex number (a + jb) is what has been previously referred to as magnitude and is defined by:

$$r = \sqrt{(a^2 + b^2)}$$

The argument is also the phase angle and is given by:

$$\tan\theta = \frac{b}{a}$$

These are the familiar forms when converting from rectangular to polar coordinates.

It is important to take great care when using the ATN function on a calculator or computer to determine the angle, because it normally only caters for angles in the range −90° to 90°.

THE ARGAND DIAGRAM

THIS IS A GRAPHICAL representation of a complex number and **Fig 4** shows how the point a + jb is plotted and how the modulus and argument are defined. This figure also shows how points in all four quadrants are represented.

In the Argand diagram the ATN function copes only with quadrants A and B. Additional lines must be inserted in the listings testing the values of a and b to cope with quadrants C and D.

ADDITIONAL PROPERTIES

COMPLEX NUMBERS HAVE a number of features, and there are four important identities which are worth knowing. These will occur in later circuit theory. If two complex numbers are to be added or subtracted it is easiest to perform it using:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$
$$\text{and } (a + jb) - (c + jd) = (a - c) + j(b - d)$$

If two complex numbers have to be multiplied or divided it may be easier to perform it by converting the complex numbers to magnitudes and angles and using the following relationships:

$$A \angle\theta_1 \times B \angle\theta_2 = A \times B \angle(\theta_1 + \theta_2)$$
$$A \angle\theta_1 / B \angle\theta_2 = A / B \angle(\theta_1 - \theta_2)$$

Where A, B and θ_1, θ_2 are respectively the modules and angle of two complex numbers.

Listings 1 to 4 can be found on page 67 ▶