# Higher Order Loops Improve Phase Noise of Feedback Oscillators

*Editor's Note: Occasionally, a paper is published that is so timeless and important, it is referenced for years. This is such a paper.*

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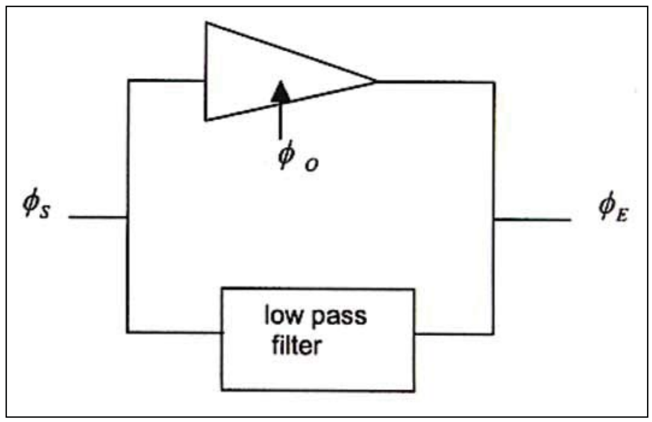
This article derives mathematical formulae for oscillators, including second- and third-order equivalent low-pass filters in their feedback loop. The mathematical models can be used to compute phase noise sidebands. Several graphs representing the phase noise characteristics of various types of oscillators are included. Numerical examples illustrate the improvement of oscillators with uncoupled resonators in their feedback loop.




▲ **Figure 1. Simple feedback oscillator model.**

## Introduction

The harmful effects of phase noise from the transmitter's exciter stage and the receiver's local oscillators were first discovered when moving target indication (MTI) was developed during World War II. Since the end of WWII, much research has been directed toward im-proving the phase noise of oscillators. Cellular systems, especially the base stations, FM transmitting and receiving techniques, the telemetric transceivers of geostationary and orbiting satellites and measuring equipment for angular modulation, all contain very low phase noise oscillators.

This research is documented in numerous articles. In 1960, W. A. Edson [1] first provided useful mathematical formula for the FM-noise deviation of linear and nonlinear oscillators. D. B. Leeson publishing an excellent paper in 1966 [2], in which he gave a heuristic solution for computing the phase noise sideband in the vicinity of the carrier frequency. Leeson's theory describes the transfer characteristic of the oscillator in the baseband, instead of the vicinity of the radio frequency (RF) carrier frequency. This frequency transposition allows determination of the flicker effect of the active (i.e., semiconductor) device. Practical oscillators corroborated Leeson's theory. A few years later, G. Sauvage [3] published an exact derivation of Leeson's formula using the theories and mathematical rules of random (stochastic) processes.

This article was inspired by the result of Sauvage's derivation, substituting higher order low-pass filter transfer functions for simple resistance-capacitance (RC) low-pass filter transfer functions. In the past few years, excellent low phase noise oscillators have been built and published using higher order filters in their feedback loop.

## Leeson's oscillator model

The simple feedback oscillator model is shown in Figure 1.

According to Sauvage, the following equation can be used:

$$\phi_S(t) + \phi_O(t) = \phi_E(t) \qquad (1)$$

$$\phi_S(t) = \int_{\infty}^{\infty} \phi_E(t) h_{BF}(t' - t)dt' = \phi_E(t) \quad h_{BF}(t) \tag{2}$$

where $h_{BF}(t)$ is the impulse response of the equivalent low-pass filter (the space in the equation denotes the convolution product), $\phi_S(t)$ converges for nearly all samples of $\phi_E(t)$. The filter is linear and time invariant, and the stationary random process $\phi_E(t)$ possesses a finite second-order moment.

Applying these conditions and using the autocorrelation function of $\phi_O(t)$ and the cross correlation functions appearing between phase fluctuations at $E$ and $S$ ports, as well as the Wiener-Khintchin theorem, Sauvage derived the following important equation:

$$S_{\phi E}(\omega) = S_{\phi O}(\omega) \cdot \left[ \left( H(j\omega) - 1 \right)\left( H^*(j\omega) - 1 \right) \right]^{-1} \tag{3}$$

The "*" denotes the conjugation process. This is Leeson's general formula for every type of equivalent low-pass filter. The $S_{\phi E}(\omega)$ is the phase noise spectrum at the output port of the oscillator as a result of the $S_{\phi O}(\omega)$ phase noise excitation. For a simple low-pass RC filter,

$$H(j\omega) = \frac{\alpha}{j\omega + \alpha}, \quad H^*(j\omega) = \frac{\alpha}{\alpha - j\omega} \tag{4}$$

where

$$\alpha = \frac{\omega_O}{2Q_L}$$

$\omega_0$ is the angular carrier (operating) frequency of the oscillator, and $Q_L$ is the loaded quality factor ($Q$) of the resonator. Equation (4) is substituted into Equation (3).

After mathematical reduction, we obtain the following formula published by Leeson:

$$S_{\phi e}(\omega) = S_{\phi O}(\omega) \left[ 1 + \left( \frac{\alpha}{\omega} \right)^2 \right] \tag{5}$$

Between the realization of Bode networks, we substitute $\alpha = \omega_i$ where:

$$\omega_i = \frac{1}{RC}$$

which gives a single real root on the horizontal axis of the complex plane.

## Second-order equivalent low-pass filter in the feedback loop

In the next step, the transfer function of the equivalent low-pass filter is substituted into Equation (3), including conjugate complex roots, according to the Bode Theory.

$$H(j\omega) = \frac{1}{1 - \left( \frac{\omega}{\omega_1} \right)^2 + j2\zeta \frac{\omega}{\omega_i}}$$
$$H^*(j\omega) = \frac{1}{1 - \left( \frac{\omega}{\omega_1} \right)^2 - j2\zeta \frac{\omega}{\omega_i}} \tag{6}$$

In this case, the "*" denotes the conjugation process. After reducing to a common denominator some further simplification and making its reciprocal, we obtain the results shown in Equation (7).

Dividing by the denominator and focusing only on the second part of the equation (omitting the exciting noise spectrum), we arrive at the next formula:

$$\left[ \left( H(j\omega) - 1 \right)\left( H^*(j\omega) - 1 \right) \right]^{-1} = 1 + \left( \frac{\omega_i}{\omega} \right)^2 \frac{1 - 2\left( \frac{\omega}{\omega_i} \right)^2}{\left( \frac{\omega}{\omega_i} \right)^2 + 4\zeta^2} \tag{8}$$

This equation inspires some idea in connection with designing of oscillators after plotting it against

$$x = \frac{\omega}{\omega_i}$$

normalized frequency, which is shown in Figure 2, with different damping ς parameters. This damping factor is taken from the theory of linear networks.

Case ς ≥ 1: Both singularities of the loop transfer function are on the real axis of the complex

$$S_{\phi E}(\omega) = S_{\phi O}(\omega) \cdot \left[ \left( H(j\omega) - 1 \right)\left( H^*(j\omega) - 1 \right) \right]^{-1} = S_{\phi O}(\omega) \cdot$$
$$\left[ \left( \frac{1}{1 - \left( \frac{\omega}{\omega_i} \right)^2 + j2\zeta \frac{\omega}{\omega_i}} - 1 \right)\left( \frac{1}{1 - \left( \frac{\omega}{\omega_i} \right)^2 - j2\zeta \frac{\omega}{\omega_i}} - 1 \right) \right]^{-1} = S_{\phi O}(\omega) \cdot \frac{1 - 2\left( \frac{\omega}{\omega_i} \right)^2 + \left( \frac{\omega}{\omega_i} \right)^4 + 4\zeta^2 \left( \frac{\omega}{\omega_i} \right)^2}{\left( \frac{\omega}{\omega_i} \right)^4 + 4\zeta^2 \left( \frac{\omega}{\omega_i} \right)^2}$$

▲ **Equation (7).**

plane and comply with loosely coupled or two uncoupled resona-tors in the radio frequency range. The larger the value of $\varsigma$ is, the better the loaded $Q$ of the resonators. According to Figure 2, the steepness of the transfer function is twice as much as that of Leeson's model (asymptotically). Figure 2 also shows the plotted transfer function of the simple RC low-pass filter.

Case $\varsigma \le 1$: Both singularities of the transfer function are on the left side of the imaginary axis in the complex plane, which represents tightly coupled resonators. In the case of

$$\varsigma = \frac{1}{\sqrt{2}}$$

maximally flat resonators are operating in the oscillator. In the case of

$$\varsigma = \frac{1}{2}$$

a peak of approximately 1.3 dB appears in the transfer function. This over-coupling generates many problems, the most severe is the abrupt jumping of the oscillator frequency versus tuning.

According to the graphs shown in Figure 2, the improvement of the phase noise is not significant and these settings can be disregarded. The loosely coupled resonators produce excellent phase noise in the near vicinity of the carrier frequency, but the available output power is low. In the majority of cases, post amplification is necessary. However, care should be taken to select a low-noise, low-distortion amplifier to produce the necessary output power. It should be noted that the intrinsic noise level of the post amplifier is also added to the phase noise level.

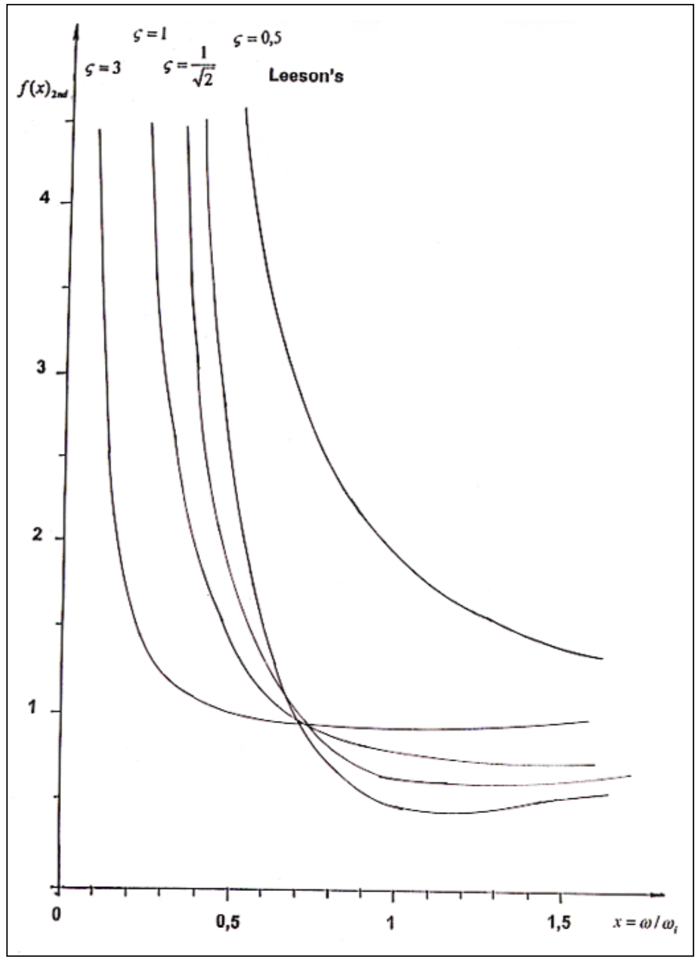
Let us now examine the case of $\varsigma > 1$, representing loosely coupled or uncoupled resonators in the oscillator's feedback loop. First we multiply two of Leeson's transfer function as follows:

$$H(j\omega) = \frac{\alpha_1}{\alpha_1 + j\omega} \cdot \frac{\alpha_2}{\alpha_2 + j\omega} = \frac{\alpha_1 \cdot \alpha_2}{\alpha_1 \cdot \alpha_2 + j\omega\alpha_1 + j\omega\alpha_2 - \omega^2} \quad (9)$$

Dividing Equation (9) by $\alpha_1\alpha_2$ yields

$$H(j\omega) = \frac{1}{1 - \dfrac{\omega^2}{\alpha_1\alpha_2} + j\omega\dfrac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2}} \quad (10)$$

When introducing




▲ **Figure 2. Second-order low-pass filter.**

$$\alpha_1\alpha_2 = \omega_i^2$$

and

$$\varsigma = \frac{\alpha_1 + \alpha_2}{2\sqrt{\alpha_1\alpha_2}}$$

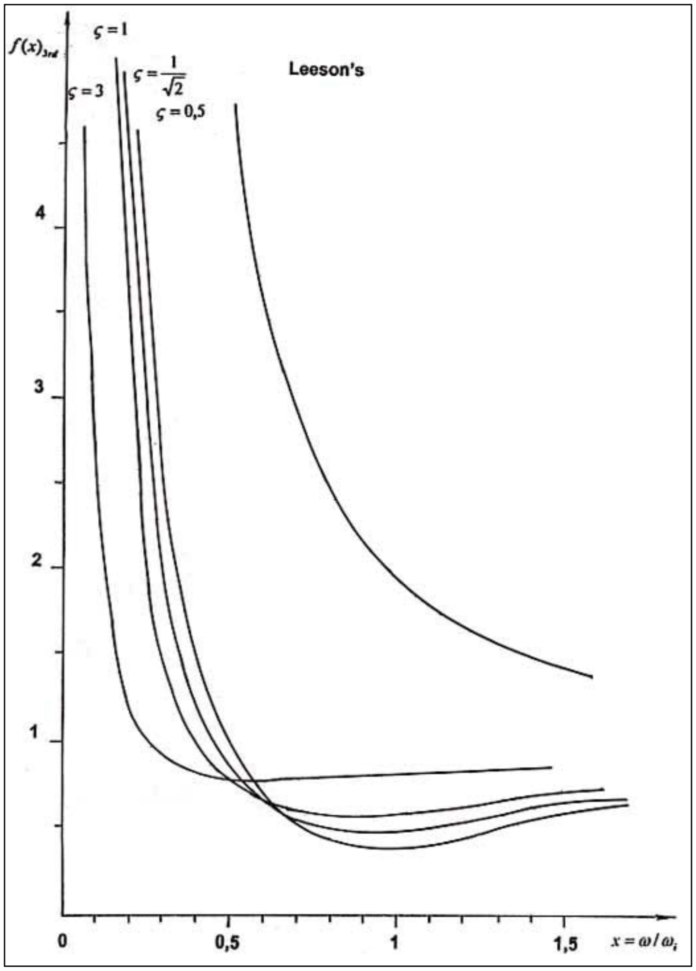
we have Equation (6).

**Third-order equivalent low-pass filter in the feedback loop**

The transfer function of the third-order equivalent low-pass filter can be generated as the multiplication of a first- and a second-order low-pass filter transfer function:

$$H(j\omega) = \frac{1}{1 - \left(\dfrac{\omega}{\omega_i}\right)^2 + j2\varsigma\dfrac{\omega}{\omega_i}} \cdot \frac{1}{1 + j\dfrac{\omega}{\omega_3}} \quad (11)$$

After executing the multiplication and introducing $\omega_3$




▲ **Figure 3. Third-order low-pass loop filter.**

$= \mu\omega_i$, we have the results shown in Equation (12), where $\mu$ is a dimensionless multiplier. Inserting Equation (12) into Equation (3), after simplification and dividing by the denominator, we obtain the third-order loop transfer function

$$\left[(H(j\omega)-1)(H^*(j\omega)-1)\right]^{-1} = 1 + \frac{1}{\left(\frac{\omega}{\omega_i}\right)^2} \cdot \frac{1-2\left(\frac{\omega}{\omega_i}\right)^2\left[1+2\frac{\varsigma}{\mu}\right]}{\left(\frac{\omega}{\omega_i}\right)^2\left[1+2\frac{\varsigma}{\mu}\right]^2+\left[2\varsigma+\frac{1}{\mu}\left\{1-\left(\frac{\omega}{\omega_i}\right)^2\right\}\right]^2} \quad (13)$$

In Figure 3, Equation (13) has been plotted against

$$x = \frac{\omega}{\omega_i}$$

$$H(j\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_i}\right)^2\cdot\left[1+2\frac{\varsigma}{\mu}\right]+j\frac{\omega}{\omega_i}\left[2\varsigma+\frac{1}{\mu}\left\{1-\left(\frac{\omega}{\omega_i}\right)^2\right\}\right]}$$

▲ **Equation (12).**

normalized frequency with different $\varsigma$ parameters. According to the plotted graphs, an additional 6 dB per octave steepness has been attained in comparison with the second-order equivalent low-pass loop filter. Similar to the second-order case, the cleaner the resultant phase noise spectrum, the larger the value of $\varsigma$ (i.e., the loaded $Q$ of the resonators). Using this model, higher order equivalent low-pass loop filters can be analyzed, and it can be proved that every increment in the degree of the loop filter produces an additional 6 dB per octave steepness enhancement asymptotically. Vidmar [4] has published an excellent low-phase noise oscillator with a third-order equivalent low-pass loop filter in the 2 to 4 GHz frequency range.

## Numerical examples

*First-order equivalent low-pass filter in the oscillator feedback loop*

This is Leeson's model with a single resonator in the feedback loop. The technical data will be used later for the higher order equivalent low-pass loop filters. The data are estimated values because the measurement is extremely difficult in some cases.

- Carrier (operating) frequency: $f_o$ = 2 GHz
- Baseband (Fourier) frequency under test: $f_m$ = 10 kHz
- Loaded $Q$ of the resonator: $Q_L$ = 10
- Noise figure of the active device: $F$ = 6 dB (numerical value: 4)
- Input power level of the device: $P_{in}$ = 1 mW
- The corner frequency of the flicker noise: $f_c$ = 3 kHz.

To make Leeson's equation more user and measurement friendly, we first derive the single sideband phase noise spectrum instead of the double sideband noise spectrum. Next, the exciting noise spectrum should be written, including the effect of the flicker noise component. As a result, we have:

$$L(f_m) = \frac{1}{2}\left[1+\frac{1}{f_m^2}\cdot\left(\frac{f_0}{2\cdot Q_t}\right)^2\right]\cdot\frac{FkT}{P_{in}}\left(1+\frac{f_c}{f_m}\right) \quad (14)$$

where $k = 1.38 \times 10^{-23}$ [Joule/Kelvin] — the Boltzmann constant, and $T$ = 293 K (at room) — absolute temperature in Kelvin. The loop transfer function is in brackets and the right expression is the exciting phase noise spectrum of the active device. The flicker noise component is shown in parenthesis. Flicker noise dominates only in the near vicinity of the carrier frequency adding a further 3 dB per octave steepness to the phase noise spectrum. Substituting our data into Equation (14), we have $L(f_m) = 1.05128 \times 10^{-9}$, or logarithmically, we have:

$$L(fm) = 10 \times \log 1.05128 \times 10^{-9} = -89.78 \text{ dB}$$
$$L(\text{fm}) \approx -90 \text{ dB}$$

at a 10 kHz deviation from the radio frequency carrier.

*Second-order equivalent low-pass filter in the oscillator feedback loop*

Let us consider an oscillator with two uncoupled resonators (i.e., Miller-oscillators), where the loaded $Q$ of the second resonator is five times as much as the first one. Using the previous data, we have:

$$\alpha_1 = \frac{\omega_0}{2Q_t} = \frac{2\pi f_0}{2Q_t} = \frac{2\pi \cdot 2 \cdot 10^9}{2 \cdot 10} = 6.283 \cdot 10^8$$

$$\alpha_2 = \frac{\alpha_1}{5} = 1.2566 \cdot 10^8$$

$$\omega_i = \sqrt{\alpha_1 \alpha_2} = \sqrt{6.283 \cdot 10^8 \cdot 1.2566 \cdot 10^8}$$
$$= 2.809 \cdot 10^8$$

$$\varsigma = \frac{6.283 + 1.2566}{2\sqrt{6.283 \cdot 1.2566}} = 1{,}3416$$

After completing the necessary mathematical operations (previously discussed), we have

$$L(f_m) = \frac{1}{2}\left[1 + \left(\frac{f_i}{f_m}\right)^2 \cdot \frac{1 - 2\left(\frac{f_m}{f_i}\right)^2}{\left(\frac{f_m}{f_i}\right)^2 + 4\varsigma^2}\right] \cdot \frac{FkT}{P_{in}}\left(1 + \frac{f_c}{f_m}\right) \quad (15)$$

where Equation (15a) is identical, with

$$\frac{f_m}{f_i} = \frac{10^4}{4.47 \cdot 10^7} = 2.237 \cdot 10^{-4} \qquad f_i = \frac{\omega_i}{2\pi} = \frac{2.809 \cdot 10^8}{2\pi} = 4.47 \cdot 10^7$$

$$L(f_m) = \frac{1}{2}\left[1 + \left(\frac{1}{2.237 \cdot 10^{-4}}\right)^2 \cdot \frac{1 - 2\left(2.237 \cdot 10^{-4}\right)^2}{\left(2.237 \cdot 10^{-4}\right)^2 + 4 \cdot 1.3416^2}\right] \cdot \frac{4 \cdot 1.38 \cdot 10^{-23} \cdot 293}{10^{-3}}\left(1 + \frac{3000}{10^4}\right)$$

$$L(f_m) = 2.918 \cdot 10^{-11}$$

▲ **Equation (15a).**

$$L(fm) = 105.35 \text{ dB}$$

phase noise level significantly improved.

*Third-order equivalent low-pass filter in the oscillator feedback loop*

Using the previous data and applying $\mu = 1$ (i.e., $\omega_3 = \omega_i$) after performing the necessary mathematical transformations and substituting our data, we have: (see Equation (16)), which is equivalent to a phase noise level of $L(fm) = 108.1\ dB$

$$L(f_m) = \frac{1}{2}\left[1 + \frac{1}{\left(\frac{f_m}{f_i}\right)^2} \cdot \frac{1 - 2\left(\frac{f_m}{f_i}\right)^2\left[1 + 2\frac{\varsigma}{\mu}\right]}{\left(\frac{f_m}{f_i}\right)^2\left[1 + 2\frac{\varsigma}{\mu}\right]^2 + \left[2\varsigma + \frac{1}{\mu}\left\{1 - \left(\frac{f_m}{f_i}\right)^2\right\}\right]^2}\right] \cdot \frac{FkT}{P_{in}}\left(1 + \frac{f_c}{f_m}\right)$$

$$= \frac{1}{2}\left[1 + \frac{1}{\left(\frac{10^4}{4.47 \cdot 10^7}\right)^2} \cdot \frac{1 - 2\left(\frac{10^4}{4.47 \cdot 10^7}\right)^2\left[1 + 2\frac{1.3416}{1}\right]}{\left(\frac{10^4}{4.47 \cdot 10^7}\right)^2\left[1 + 2\frac{1.3416}{1}\right]^2 + \left[2 \cdot 1.3416 + \left\{1 - \left(\frac{10^4}{4.47 \cdot 10^7}\right)^2\right\}\right]^2}\right]$$

$$\cdot \frac{4 \cdot 1.38 \cdot 10^{-23} \cdot 293}{10^{-3}} \cdot \left(1 + \frac{3000}{10^4}\right) = 1.5484 \cdot 10^{-11}$$

▲ **Equation (16).**

## Conclusion

The presented mathematical models derived with simplified conditions (i.e., the system) are perfectly linear. This condition is not true because the active device is nonlinear, at its input and output ports clipped (limited) sines wave can be found, so it works in conduction angle mode of operation. At the output of high *Q* resonators, the signal is quasi-sinusoidal, but at the output port of the active device, the signal is distorted. Determining the noise factor of an active device operating in conduction angle condition and measuring and calculating the loaded *Q* of an operating oscillator are difficult problems.

In spite of these difficulties, it is worthwhile to calculate the phase noise spectrum level, taking into consideration the following hints:

- Input level may be determined from the output level using the amplification factor for large signals.
- The loaded *Q* can be roughly measured by adding extra parallel loads to the resonator in experimental (sample) oscillators. (In the gigahertz range, that can be measured by the standing wave method with certain conditions and limitations).
- The conduction angle might be measured on operating experimental oscillator models (in the kilohertz or megahertz frequency range to obstain exact measurements).
- The effective noise factor might also be calculated with the help of the previously determined conduction angle.
- The required accuracy might be realized by employing successive approximation corrections.

In practice, it is difficult to fit a tiny active device into a resonator or coupled resonators. Special care should be taken to realize the correct phase shifts. The resonators are stripline or microstrip printed circuits fabricated on fiberglass-polyester, Teflon, ceramic-filled Teflon and ceramic materials. For the sake of better Q, the resonators are printed on double-sided ceramic material covered by another one-sided copper-clad plate, and the lower and upper conductive plates are connected by several conductive "via-holes" forming conductive walls. This arrangement minimizes the radiation losses resulting in better *Q*. These oscillators are available from many manufacturers "off the shelf," realized in an SMD case.

The oscillator design can be aided by software, especially the scattering parameter and layout effects. ■

## References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